

APPROXIMATION OF SUBCATEGORIES BY ABELIAN SUBCATEGORIES.

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ABSTRACT. Let \mathcal{C} be an abelian category and let \mathcal{D} be an additive full subcategory of \mathcal{C} . We prove a useful criterion for deciding whether \mathcal{D} is itself abelian. When \mathcal{D} is not abelian, we give a construction for producing a “best reflective abelian approximation” to \mathcal{D} . We describe applications of this construction to I -adic completion of modules over commutative rings (in which case we recover the category of “ L -complete modules” studied by Greenlees-May and Hovey-Strickland) and I -adic completion of comodules over Hopf algebroids.

1. INTRODUCTION.

Let \mathcal{C} be an abelian category, and let \mathcal{D} be a full subcategory of \mathcal{C} . Suppose \mathcal{D} has finite limits and colimits. One wants to know that \mathcal{D} is an abelian category, and one wants to know whether kernels and cokernels in \mathcal{D} agree with kernels and cokernels taken in \mathcal{C} . Of course, we are not always so lucky: for example, let R be a commutative topological ring which is preadic (see [4] for definitions). Then the category $\text{CHMod}(R)$ of complete Hausdorff R -modules (in the I -adic topology, where I is any ideal of definition of R) is a subcategory of the category $\text{Mod}(R)$ of R -modules, and $\text{CHMod}(R)$ has all its finite limits, but the cokernel of a morphism $M \xrightarrow{f} N$ considered as a morphism in $\text{CHMod}(R)$ does not necessarily agree with the cokernel of f considered as a morphism in $\text{Mod}(R)$ (see [9] for details). The category $\text{CHMod}(R)$ also fails to be an abelian category, since images and coimages of morphisms in $\text{CHMod}(R)$ do not always agree (see [9] for details).

The purpose of this note is to describe a useful criterion (Prop. 2.5) for detecting whether a full subcategory of an abelian category \mathcal{C} is an abelian category whose kernels and cokernels agree with those in \mathcal{C} ; to describe some of the structure (see e.g. Prop. 3.1) that the full subcategory inherits from \mathcal{D} ; and we construct, in Prop. 4.6, a “best reflective abelian approximation” for certain full subcategories of an abelian category. Our “best reflective abelian approximation” construction, when applied to the I -adic completion functor on the category of R -modules, produces the category of R -modules M such that the functor $M \rightarrow L_0F(M)$ is an isomorphism, where L_0F is the zeroth left derived functor of I -adic completion. This category, which is a generalization of the “Ext- p -complete groups” of Bousfield and Kan in [1], was studied by Greenlees and May, in [3], as a substitute for the category of complete Hausdorff R -modules; our Prop. 4.6 tells us that this category of Greenlees and May is not an isolated or ad hoc construction, but really is the abelian category which is the “best reflective abelian approximation” to

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the category of complete Hausdorff R -modules. Some applications of Prop. 4.6 are described in Cor. 4.7 and in the future papers [9] and [8].

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2. ABELIAN SUBCATEGORIES.

In this section we prove some basic facts about subcategories of abelian categories, comparable to those in [2]. The purpose of this section is Prop. 2.5, in which we prove a criterion for deciding whether an additive full subcategory of an abelian category is itself abelian.

Lemma 2.1. Full subcategories of abelian categories which are closed under kernels, cokernels, finite products, and finite coproducts are abelian.

Let \mathcal{C} be an abelian category, and let \mathcal{D} be a full subcategory of \mathcal{C} such that:

- *if $M \xrightarrow{f} N$ is a morphism in \mathcal{D} , then the kernel $\ker f$ and cokernel $\operatorname{coker} f$, computed in \mathcal{C} , are contained in \mathcal{D} , and*
- *if M, N are two objects of \mathcal{D} , then the biproduct $M \oplus N$, computed in \mathcal{C} , is contained in \mathcal{D} .*

Then \mathcal{D} is an abelian category.

Proof. Let $M \xrightarrow{f} N$ be a morphism in \mathcal{D} , and let $T \xrightarrow{g} M$ be a morphism in \mathcal{D} such that $f \circ g = 0$. Then g factors uniquely through the kernel $\ker f$ in \mathcal{C} , and since $\ker f$ is (by assumption) an object of \mathcal{D} , kernels exist in \mathcal{D} and agree with the kernels computed in \mathcal{C} . A precisely dual argument shows that cokernels in \mathcal{D} exist and agree with cokernels computed in \mathcal{C} .

If $M \xrightarrow{f} N$ is a monomorphism in \mathcal{D} , then $\ker(\operatorname{coker} f) = f$, with the kernels and cokernels taken in \mathcal{C} , since \mathcal{C} is an abelian category; and since kernels and cokernels in \mathcal{D} agree with those in \mathcal{C} , we have that $\ker(\operatorname{coker} f) = f$ with the kernels and cokernels taken in \mathcal{D} instead. A precisely dual argument shows that $\operatorname{coker}(\ker f) = f$ for any epimorphism f in \mathcal{D} .

Similarly, let $\{M_i\}_{i \in I}$ be a finite set of objects in \mathcal{D} , and let $T \xrightarrow{f_i} M_i$ be a morphism in \mathcal{D} for each $i \in I$. Then each f_i factors uniquely through the product $\prod_{i \in I} M_i$ in \mathcal{C} , and since $\prod_{i \in I} M_i$ is (by assumption) an object of \mathcal{D} , finite products exist in \mathcal{D} and agree with finite products computed in \mathcal{C} . A precisely dual argument shows that finite coproducts in \mathcal{D} exist and agree with finite coproducts computed in \mathcal{C} . \square

Corollary 2.2. *Let \mathcal{C} be an abelian category and let \mathcal{D} be a full subcategory of \mathcal{C} such that the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$ has an exact left (resp. right) adjoint $F : \mathcal{C} \rightarrow \mathcal{D}$. Then \mathcal{D} is an abelian category.*

This is not general enough for our needs. We would like to know whether the full subcategory of \mathcal{C} generated by the objects FX is abelian, when F is left exact or right exact, but not both.

Lemma 2.3. Kernels and cokernels in a (co)reflective subcategory. *Let \mathcal{C} be an abelian category and let \mathcal{D} be a full subcategory of \mathcal{C} such that the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$ has a left (resp. right) adjoint $\mathcal{C} \xrightarrow{F} \mathcal{D}$. Then, for any morphism $M \xrightarrow{f} N$, f has a kernel (resp. cokernel) in \mathcal{D} , and it is naturally isomorphic to*

the kernel (resp. cokernel) of f in \mathcal{C} ; and f has a cokernel (resp. kernel) in \mathcal{D} , and it is naturally isomorphic to F applied to the cokernel (resp. kernel) in \mathcal{C} .

Proof. Assume F is left adjoint to the forgetful functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$; the dual case is handled dually. We will write \ker and coker for kernels and cokernels computed in \mathcal{C} . Let $T \xrightarrow{g} M$ be a morphism in \mathcal{D} such that $f \circ g = 0$. Then $Gf \circ Gg = 0$, so Gg factors uniquely through $\ker Gf$, and since G is a right adjoint, hence preserves limits and in particular equalizers, Gg factors uniquely through $G \ker f \cong \ker Gf$. We know that FG is naturally isomorphic to the identity functor on \mathcal{D} , since \mathcal{D} is a full subcategory of \mathcal{C} ; so we apply F to the factorization of Gg through $G \ker f$ to get that g factors uniquely through $\ker f$. Hence $\ker f$ exists in \mathcal{D} and is the kernel of f .

Now let $N \xrightarrow{g'} T'$ be a morphism in \mathcal{D} such that $g' \circ f = 0$. Then Gg' factors uniquely through $\operatorname{coker} Gf$, and since T' is in \mathcal{D} , the map $\operatorname{coker} Gf \rightarrow T'$ factors uniquely through $F \operatorname{coker} Gf$; so the morphism f has a cokernel in \mathcal{D} , and that cokernel is $F \operatorname{coker} Gf$. \square

Recall that a subcategory \mathcal{D} of a category \mathcal{C} is said to be a *reflective subcategory* if there exists a left adjoint to the forgetful functor $\mathcal{D} \rightarrow \mathcal{C}$. This left adjoint is sometimes called the *reflector functor*. Dually, \mathcal{D} is said to be a *coreflective subcategory* if there exists a right adjoint to the forgetful functor, and this right adjoint is called the *coreflector functor*.

Corollary 2.4. *Let \mathcal{C} be an abelian category and let \mathcal{D} be a full subcategory of \mathcal{C} such that the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$ has a left (resp. right) adjoint $\mathcal{C} \xrightarrow{F} \mathcal{D}$. Then any morphism in \mathcal{D} is monic (resp. epic) in \mathcal{D} iff it is monic (resp. epic) in \mathcal{C} . Furthermore, if a morphism f in \mathcal{D} is epic (resp. monic) in \mathcal{C} , then it is epic (resp. monic) in \mathcal{D} .*

When \mathcal{D} is a full subcategory of an abelian category \mathcal{C} such that the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint, monics in \mathcal{D} are precisely monics in \mathcal{C} , but knowing that a map is epic in \mathcal{D} is generally not enough to conclude that it is epic in \mathcal{C} . We could ask about what happens when being epic in \mathcal{D} is enough to know that a morphism is also epic in \mathcal{C} . It turns out that this is a necessary condition for \mathcal{D} to be an abelian category:

Proposition 2.5. Criterion for being an abelian subcategory. *Let \mathcal{C} be an abelian category and let \mathcal{D} be a full subcategory of \mathcal{C} such that the inclusion functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ has a left (resp. right) adjoint F . Then the following are equivalent:*

- (1) \mathcal{D} is an abelian category.
- (2) For every morphism $M \xrightarrow{f} N$ in \mathcal{D} , the map $\operatorname{coker} Gf \rightarrow G \operatorname{coker} f$ is monic in \mathcal{C} (resp. the map $G \ker f \rightarrow \ker Gf$ is epic in \mathcal{C}).
- (3) For every morphism $M \xrightarrow{f} N$ in \mathcal{D} , the map $\operatorname{coker} Gf \rightarrow GF \operatorname{coker} Gf$ is monic in \mathcal{C} (resp. the map $GF \ker Gf \rightarrow \ker Gf$ is epic in \mathcal{C}).

Proof. We handle the case when F is left adjoint to G ; the dual case is handled dually.

- Condition 2 is equivalent to condition 1:

Let $M \xrightarrow{f} N$ be monic in \mathcal{D} . Then, for \mathcal{D} to be an abelian category, f must be the kernel of its cokernel. We have the commutative diagram in \mathcal{C}

with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & GM & \xrightarrow{Gf} & GN & \xrightarrow{g} & \text{coker } Gf \longrightarrow 0 \\
 & & \downarrow \text{id}_{GM} & & \downarrow \text{id}_{GN} & & \downarrow \eta \\
 0 & \longrightarrow & GM & \xrightarrow{Gf} & GN & \longrightarrow & G \text{ coker } f
 \end{array}$$

and the exact sequence (which holds for any composable pair of morphisms in any abelian category)

$$0 \rightarrow \ker g \rightarrow \ker \eta \circ g \rightarrow \ker \eta \rightarrow \text{coker } g$$

is isomorphic to the extension in \mathcal{C}

$$0 \rightarrow \ker g \rightarrow \ker \eta \circ g \rightarrow \ker \eta \rightarrow 0$$

since g is an epimorphism in \mathcal{C} . Now the map $\eta \circ g$ is the cokernel map of f in \mathcal{D} , so f is the kernel of its cokernel if and only if the map $M \rightarrow \ker \eta \circ g$ is an isomorphism; and this is true if and only if $\ker \eta \cong 0$ in \mathcal{C} , i.e., if and only if η is monic in \mathcal{C} .

Now suppose $M \xrightarrow{f} N$ is epic in \mathcal{D} , and suppose that monomorphisms in \mathcal{D} are the kernels (in \mathcal{D}) of their cokernels (in \mathcal{D}), i.e., that for every object X of \mathcal{C} , the map $X \rightarrow GFX$ is monic in \mathcal{D} . The map f is epic in \mathcal{D} if and only if $GF \text{ coker } f \cong 0$, and since $\text{coker } f$ injects into $GF \text{ coker } f$, the map f is epic in \mathcal{D} if and only if it is epic in \mathcal{C} . When f is epic in \mathcal{C} then f is the cokernel of its kernel in \mathcal{D} since in that case the kernel and cokernel agree with those in \mathcal{C} , and \mathcal{C} is an abelian category (hence every epimorphism is the cokernel of its kernel in \mathcal{C}).

- *Condition 2 is equivalent to condition 3:* The cokernel $G \text{ coker } f$ is naturally isomorphic to $GF \text{ coker } Gf$, by Lemma 2.3.

□

Lemma 2.6. Limits and colimits in a (co)reflective abelian subcategory.

Let \mathcal{C} be an abelian category and let \mathcal{D} be a full subcategory of \mathcal{C} such that the inclusion functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ has a left (resp. right) adjoint F and such that \mathcal{D} is an abelian category.

- If X is a small diagram in \mathcal{D} such that the diagram GX has a colimit (resp. limit) in \mathcal{C} , then X has a colimit (resp. limit) in \mathcal{D} , and it is naturally isomorphic to $F \text{ colim } GX$ (resp. $F \text{ lim } GX$).
- If X is a finite diagram in \mathcal{D} such that the diagram GX has a limit (resp. colimit) in \mathcal{C} , then X has a limit (resp. colimit) in \mathcal{D} , and it is naturally isomorphic to $F \text{ lim } GX$ (resp. $F \text{ colim } GX$).

Proof. Suppose F is left adjoint to G (the dual case is handled dually). Then F preserves colimits, so $\text{colim } GX$ is the terminal object in cones over the diagram GX , and $F \text{ colim } GX$ is the terminal object in cones over the diagram FGX ; but FG is naturally isomorphic to the identity functor on \mathcal{D} , since \mathcal{D} is a full subcategory of \mathcal{C} . So $F \text{ colim } GX$ is the terminal object in cones over the diagram X , i.e., $\text{colim } X \cong F \text{ colim } GX$. If X is a finite diagram, then $\text{lim } X$ exists in \mathcal{D} because \mathcal{D}

is an abelian category and hence has all its finite limits; furthermore, G is a right adjoint, so G preserves small limits, and

$$F \lim GX \cong FG \lim X \cong \lim X.$$

□

Corollary 2.7. *Let \mathcal{C} be an abelian category and let \mathcal{D} be a full subcategory of \mathcal{C} such that the inclusion functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ has a left (resp. right) adjoint F and such that \mathcal{D} is an abelian category. Suppose that \mathcal{C} is co-complete (resp. complete), i.e., that \mathcal{C} has all small colimits (resp. all small limits). Then \mathcal{D} is co-complete (resp. complete).*

3. MONOIDAL STRUCTURES.

In this section we prove that, when \mathcal{C} is a closed monoidal abelian category, any subcategory of \mathcal{C} which is an abelian category inherits a closed monoidal structure from \mathcal{C} ; if \mathcal{C} is braided or symmetric then any abelian subcategory of \mathcal{C} is, as well.

Proposition 3.1. *Let \mathcal{C} be a closed monoidal abelian category, Let \mathcal{D} be a full subcategory of \mathcal{C} which is an abelian category and such that the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint $c \xrightarrow{F} \mathcal{D}$.*

- \mathcal{D} is a closed monoidal abelian category, with monoidal product $X \otimes_{\mathcal{D}} Y$ of $X, Y \in \text{ob } \mathcal{D}$ given by

$$F(X) \otimes_{\mathcal{D}} F(Y) \cong F(X \otimes_{\mathcal{C}} Y),$$

and with unit object $1_{\mathcal{D}}$ isomorphic to $F(1_{\mathcal{C}})$. With this monoidal product on \mathcal{D} , the functor F is strong monoidal, and G is lax monoidal. The internal hom in \mathcal{D} agrees with that of \mathcal{C} , i.e., if X, Y are objects of \mathcal{D} , then $[X, Y]_{\mathcal{C}}$ is also an object of \mathcal{D} , and we let $[X, Y]_{\mathcal{D}}$ be the internal hom in \mathcal{D} as well as in \mathcal{C} .

- If \mathcal{C} is braided monoidal, let

$$X \otimes_{\mathcal{C}} Y \xrightarrow{\chi_{X,Y}} Y \otimes_{\mathcal{C}} X$$

be the braiding isomorphism, natural in X and Y ; then we have a natural braiding on \mathcal{D} given as the composite

$$\begin{array}{ccc} X \otimes_{\mathcal{D}} Y & & Y \otimes_{\mathcal{D}} X \\ \downarrow \cong & & \cong \uparrow \\ F(GX \otimes_{\mathcal{C}} GY) & \xrightarrow{F\chi_{GX,GY}} & F(GY \otimes_{\mathcal{C}} GX). \end{array}$$

- If \mathcal{C} is symmetric monoidal—i.e., \mathcal{C} is braided monoidal and the composite $\chi_{Y,X} \circ \chi_{X,Y}$ is equal to the identity morphism on $X \otimes_{\mathcal{C}} Y$ —then the braiding on \mathcal{D} defined above makes \mathcal{D} symmetric monoidal.

Proof. • We first check that F will be strong monoidal, i.e., that the natural morphism

$$F(X \otimes_{\mathcal{C}} Y) \rightarrow F(GF(X) \otimes_{\mathcal{C}} GF(Y)) \xrightarrow{\cong} F(X) \otimes_{\mathcal{D}} F(Y)$$

is an isomorphism in \mathcal{D} . This will follow from showing that the natural morphism $F(X \otimes_{\mathcal{C}} Y) \rightarrow F(X \otimes_{\mathcal{C}} GF(Y))$ is an isomorphism. Let $\Psi_X : \mathcal{C} \rightarrow \mathcal{C}$ be the functor $Y \rightarrow X \otimes_{\mathcal{C}} Y$. Then FY is the initial object in the

category of objects of \mathcal{D} equipped with a map from Y ; we will write $\{Y/\mathcal{D}\}$ for this category, and $\lim\{Y/\mathcal{D}\}$ for its initial object. Then every map from $\Psi_X Y$ to an object in $\{\Psi_X Y/\mathcal{D}\}$ factors through $\Psi_X \lim\{Y/\mathcal{D}\}$, so we have maps

$$\begin{array}{ccc} \Psi_X Y & & \\ \downarrow & & \\ \Psi_X FY & \xrightarrow{\cong} & \Psi_X \lim\{Y/\mathcal{D}\} \\ \downarrow & & \\ F\Psi_X Y & \xrightarrow{\cong} & \lim\{\Psi_X Y/\mathcal{D}\}, \end{array}$$

where the composite of the two vertical maps is the natural map $\Psi_X Y \rightarrow F\Psi_X Y$ given by F being left adjoint to the full, faithful functor G . After applying F to the above diagram, we get the maps

$$F\Psi_X Y \rightarrow F\Psi_X FY \rightarrow FF\Psi_X Y \xrightarrow{\cong} F\Psi_X Y$$

whose composite is an isomorphism. Similarly, the map $\Psi_X FY \rightarrow F\Psi_X FY$ factors through $F\Psi_X Y$:

$$\begin{array}{ccc} \Psi_X FY & \xrightarrow{\cong} & \Psi_X \lim\{Y/\mathcal{D}\} \\ \downarrow & & \\ F\Psi_X Y & \xrightarrow{\cong} & \lim\{\Psi_X Y/\mathcal{D}\} \\ \downarrow & & \\ F\Psi_X FY & \xrightarrow{\cong} & F\Psi_X \lim\{Y/\mathcal{D}\}, \end{array}$$

and on applying F we get the maps

$$F\Psi_X FY \rightarrow FF\Psi_X Y \rightarrow FF\Psi_X FY \xrightarrow{\cong} F\Psi_X FY$$

whose composite is an isomorphism. Hence the natural map $F\Psi_X Y \rightarrow F\Psi_X FY$ is an isomorphism, i.e., the natural map $F(X \otimes_c Y) \rightarrow F(X \otimes_c GFY)$ is an isomorphism. We repeat this same argument using a functor

$$\begin{array}{ccc} \Phi_{GFY} : \mathcal{C} & \rightarrow & \mathcal{C} \\ X & \mapsto & X \otimes_c GFY \end{array}$$

in place of Ψ_X to prove that the map $F(X \otimes_c GFY) \rightarrow F(GFX \otimes_c GFY)$ is also an isomorphism; hence F is strong monoidal.

We will use the associator and unitor isomorphisms of \mathcal{C} to construct associator and unitor isomorphisms for \mathcal{D} . Recall that, given objects X, Y, Z in \mathcal{C} , the associator is an isomorphism

$$(X \otimes_c Y) \otimes_c Z \xrightarrow{\alpha_{X,Y,Z}} X \otimes_c (Y \otimes_c Z),$$

and for any 4-tuple W, X, Y, Z of objects in \mathcal{C} , we require the diagram

$$(3.1) \quad \begin{array}{ccc} ((W \otimes_{\mathcal{C}} X) \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z & \xrightarrow{\alpha_{W \otimes_{\mathcal{C}} X, Y, Z}} & (W \otimes_{\mathcal{C}} X) \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z) \\ \downarrow \alpha_{W, X, Y \otimes_{\mathcal{C}} \text{id}_Z} & & \downarrow \alpha_{W, X, Y \otimes_{\mathcal{C}} Z} \\ (W \otimes_{\mathcal{C}} (X \otimes_{\mathcal{C}} Y)) \otimes_{\mathcal{C}} Z & & \\ \downarrow \alpha_{W, X \otimes_{\mathcal{C}} Y, Z} & & \\ W \otimes_{\mathcal{C}} ((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{\text{id}_W \otimes_{\mathcal{C}} \alpha_{X, Y, Z}} & W \otimes_{\mathcal{C}} (X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) \end{array}$$

to commute. Continuing to write $\alpha_{X, Y, Z}$ for the associator in \mathcal{C} , we define an associator in \mathcal{D} as the composite

$$\begin{array}{ccc} (X \otimes_{\mathcal{D}} Y) \otimes_{\mathcal{D}} Z & & X \otimes_{\mathcal{D}} (Y \otimes_{\mathcal{D}} Z) \\ \downarrow \cong & & \uparrow \cong \\ F(GF(GX \otimes_{\mathcal{C}} GY) \otimes_{\mathcal{C}} GZ) & & F(GX \otimes_{\mathcal{C}} GF(GY \otimes_{\mathcal{C}} GZ)) \\ \downarrow \cong & & \uparrow \cong \\ F((GX \otimes_{\mathcal{C}} GY) \otimes_{\mathcal{C}} GZ) & \xrightarrow{F\alpha_{GX, GY, GZ}} & F(GX \otimes_{\mathcal{C}} (GY \otimes_{\mathcal{C}} GZ)). \end{array}$$

Verification that the associator in \mathcal{D} makes the analogue in \mathcal{D} of the diagram 3.1 commute follows immediately from every object in the diagram being naturally isomorphic to F applied to an object in diagram 3.1, e.g.

$$\begin{aligned} ((W \otimes_{\mathcal{D}} X) \otimes_{\mathcal{D}} Y) \otimes_{\mathcal{D}} Z &\cong F(GF(GF(GW \otimes_{\mathcal{C}} GX) \otimes_{\mathcal{C}} GY) \otimes_{\mathcal{C}} GZ) \\ &\cong F(((GW \otimes_{\mathcal{C}} GX) \otimes_{\mathcal{C}} GY) \otimes_{\mathcal{C}} GZ). \end{aligned}$$

Similarly, the left and right unitors for an object X of \mathcal{C} are isomorphisms

$$X \otimes_{\mathcal{C}} 1_{\mathcal{C}} \xrightarrow{\rho_X} X$$

and

$$1_{\mathcal{C}} \otimes_{\mathcal{C}} X \xrightarrow{\lambda_X} X$$

making the diagram

$$(3.2) \quad \begin{array}{ccc} (X \otimes_{\mathcal{C}} 1_{\mathcal{C}}) \otimes_{\mathcal{C}} Y & \xrightarrow{\alpha_{X, 1_{\mathcal{C}}, Y}} & X \otimes_{\mathcal{C}} (1_{\mathcal{C}} \otimes_{\mathcal{C}} Y) \\ & \searrow \rho_X \otimes_{\mathcal{C}} \text{id}_Y & \swarrow \text{id}_X \otimes_{\mathcal{C}} \lambda_Y \\ & X \otimes_{\mathcal{C}} Y & \end{array}$$

commute. Continuing to write ρ_X for the right unitor in \mathcal{C} , we define a right unitor in \mathcal{D} as the composite of the isomorphisms

$$\begin{array}{ccc} X \otimes_{\mathcal{D}} 1_{\mathcal{D}} & & X \\ \downarrow \cong & & \uparrow \cong \\ F(GX \otimes_{\mathcal{C}} GF1_{\mathcal{C}}) & & \\ \downarrow \cong & & \\ F(GX \otimes_{\mathcal{C}} 1_{\mathcal{C}}) & \xrightarrow{F\rho_{GX}} & FGX, \end{array}$$

and similarly for the left unitor in \mathcal{D} . Verification that these definitions of ρ and λ in \mathcal{D} make the analogue of diagram 3.2 in \mathcal{D} commute is routine.

- To show that \mathcal{D} is braided monoidal with the braiding isomorphism given in the statement of the proposition—for which we will write $\chi_{X,Y}^{\mathcal{D}}$ —we only need to show that the diagrams

$$\begin{array}{ccc}
 & (X \otimes_{\mathcal{D}} Y) \otimes_{\mathcal{D}} Z & \\
 \chi_{X,Y}^{\mathcal{D}} \otimes_{\mathcal{D}} \text{id}_Z \swarrow & & \searrow \alpha_{X,Y,Z} \\
 (Y \otimes_{\mathcal{D}} X) \otimes_{\mathcal{D}} Z & & X \otimes_{\mathcal{D}} (Y \otimes_{\mathcal{D}} Z) \\
 \alpha_{Y,X,Z} \downarrow & & \downarrow \chi_{X,Y \otimes_{\mathcal{D}} Z}^{\mathcal{D}} \\
 Y \otimes_{\mathcal{D}} (X \otimes_{\mathcal{D}} Z) & & (Y \otimes_{\mathcal{D}} Z) \otimes_{\mathcal{D}} X \\
 \text{id}_Y \otimes_{\mathcal{D}} \chi_{X,Z}^{\mathcal{D}} \searrow & & \swarrow \alpha_{Y,Z,X} \\
 & Y \otimes_{\mathcal{D}} (Z \otimes_{\mathcal{D}} X) &
 \end{array}$$

and

$$\begin{array}{ccc}
 & X \otimes_{\mathcal{D}} (Y \otimes_{\mathcal{D}} Z) & \\
 \text{id}_X \otimes_{\mathcal{D}} \chi_{Y,Z}^{\mathcal{D}} \swarrow & & \searrow \alpha_{X,Y,Z}^{-1} \\
 X \otimes_{\mathcal{D}} (Z \otimes_{\mathcal{D}} Y) & & (X \otimes_{\mathcal{D}} Y) \otimes_{\mathcal{D}} Z \\
 \alpha_{X,Z,Y}^{-1} \downarrow & & \downarrow \chi_{X \otimes_{\mathcal{D}} Y, Z}^{\mathcal{D}} \\
 (X \otimes_{\mathcal{D}} Z) \otimes_{\mathcal{D}} Y & & Z \otimes_{\mathcal{D}} (X \otimes_{\mathcal{D}} Y) \\
 \chi_{X,Z}^{\mathcal{D}} \otimes_{\mathcal{D}} \text{id}_Y \searrow & & \swarrow \alpha_{Z,X,Y}^{-1} \\
 & (Z \otimes_{\mathcal{D}} X) \otimes_{\mathcal{D}} Y &
 \end{array}$$

commute. This verification is routine, following immediately from arguments given in the proof of the previous part of this proposition.

Finally, we must show that, if X, Y are in \mathcal{D} , then so is $[X, Y]_c$, and with this internal hom, \mathcal{D} is closed monoidal. If we fix Y , then using e.g. chapter X section 1 of [7], we know that we can compute $[X, Y]_c$ as a colimit:

$$[X, Y]_c \cong \text{colim}\{Z : Z \otimes_c Y \rightarrow X\},$$

where the expression on the right-hand side means the terminal object in the category of objects Z of \mathcal{C} equipped with a morphism $Z \otimes_c Y \rightarrow X$. Now since X is in \mathcal{D} , any map $Z \otimes_c Y \rightarrow X$ factors through

$$Z \otimes_c Y \rightarrow F(Z \otimes_c Y) \xrightarrow{\cong} F(F(Z) \otimes_c Y),$$

and the map $Z \otimes_c Y \rightarrow F(F(Z) \otimes_c Y)$ factors through $Z \otimes_c Y \rightarrow F(Z) \otimes_c Y$; so the terminal object in $\text{colim}\{Z : Z \otimes_c Y \rightarrow X\}$, i.e., $[X, Y]_c$, is in \mathcal{D} .

We must show that $X \mapsto [X, Y]_c$ is right adjoint to $X \mapsto Y \otimes_{\mathcal{D}} X$, if Y is in \mathcal{D} . We have the sequence of isomorphisms:

$$\begin{aligned} \mathrm{hom}_{\mathcal{D}}(Y \otimes_{\mathcal{D}} X, Z) &\cong \mathrm{hom}_{\mathcal{D}}(F(GY \otimes_c GX), Z) \\ &\cong \mathrm{hom}_c(GY \otimes_c GX, GZ) \\ &\cong \mathrm{hom}_c(GY, [GX, GZ]_c) \\ &\cong \mathrm{hom}_{\mathcal{D}}(Y, [X, Z]_c). \end{aligned}$$

- We must simply show that, for any objects X, Y in \mathcal{D} , the composite $\chi_{Y,X}^{\mathcal{D}} \circ \chi_{X,Y}^{\mathcal{D}}$ is equal to the identity morphism on $X \otimes_{\mathcal{D}} Y$. This follows from the commutativity of the diagram

$$\begin{array}{ccccc} X \otimes_{\mathcal{D}} Y & \xrightarrow{\chi_{X,Y}^{\mathcal{D}}} & Y \otimes_{\mathcal{D}} X & \xrightarrow{\chi_{Y,X}^{\mathcal{D}}} & X \otimes_{\mathcal{D}} Y \\ \downarrow = & & \downarrow = & & \downarrow = \\ F(GX \otimes_c GY) & \xrightarrow{F(\chi_{GX,GY})} & F(GY \otimes_c GX) & \xrightarrow{F(\chi_{GY,GX})} & F(GX \otimes_c GY) \end{array}$$

as well as the fact that

$$F(\chi_{GY,GX} \circ \chi_{GX,GY}) = F(\mathrm{id}_{GX \otimes_c GY}) = \mathrm{id}_{F(GX \otimes_c GY)}.$$

□

4. CATEGORIES OF L_0F -COMPLETE AND R^0F -COMPLETE OBJECTS.

In this section we prove an abelian approximation theorem (Prop. 4.6) for certain full subcategories of abelian categories. More precisely, let \mathcal{C} be an abelian category, and let $\mathcal{C} \xrightarrow{F} \mathcal{C}$ be an additive idempotent functor. If the additive functor F is neither left exact nor right exact, then it is not necessarily the case that the essential image of F is an abelian category; for example, if \mathcal{C} is the category of R -modules, where R is an adic ring (as in [4]), and F is the I -adic completion functor $M \rightarrow \lim_{j \rightarrow \infty} M/I^j M$, where I is any ideal of definition of R , then the essential image of F is not an abelian category for most choices of R , i.e., the category of complete Hausdorff R -modules is typically not abelian (see [9] for a precise statement, proofs, and discussion). Nevertheless, one wants to have an abelian category which is a subcategory of \mathcal{C} and which is a “best abelian approximation” to the essential image of F . For example, when \mathcal{C} is the category of R -modules and F is I -adic completion, one wants to know that there exists an abelian category of (not necessarily finitely generated) R -modules, which is a “best approximation” to the category of complete Hausdorff R -modules.

The purpose of this section is Prop. 4.6, in which we provide two solutions to this problem of constructing a “best abelian approximation” to the essential image of F , a “reflective” solution and a “coreflective” solution (and in the paper [9], we show that Ext groups in these abelian categories are suitably well-behaved).

Remark 4.1. This approach toward constructing a reasonable category of not-necessarily-finitely-generated R -modules which is a good approximation to the category of complete Hausdorff R -modules is essentially due to [3] and developed further by [6]; in [9] we present a case for the category of L_0F -complete modules being the “correct” category of quasicohherent (but not necessarily coherent!) R -modules for the purposes of commutative algebra over affine formal schemes, and in

later papers, we use these constructions extensively to do commutative algebra and homological algebra (e.g. constructing Cousin complexes, in [8]) over quasicompact formal schemes, adic Hopf algebroids, and Artin stacks fibered in groupoids over formal schemes; an example of the latter kind is the moduli stack of one-dimensional formal A -modules, where A is a p -adic number ring. In a forthcoming paper we use the Cousin complex over this stack (which is fibered in groupoids over formal schemes) to make some computations of its flat cohomology, which we then apply to the computation of the E_2 -term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres.

Lemma 4.2. *Let \mathcal{C} be an abelian category with enough projectives (resp. enough injectives). Let $\mathcal{C} \xrightarrow{F} \mathcal{C}$ be an idempotent (i.e., there exists a natural equivalence $F \circ F \xrightarrow{\sim} F$) additive functor. Then there exists a natural equivalence of δ -functors $L_*(F \circ F) \xrightarrow{\sim} L_*F$ (resp. $R^*(F \circ F) \xrightarrow{\sim} R^*F$).*

Proof. See [5] for definitions and basic properties of δ -functors. Let \mathcal{C} have enough projectives; the dual case is handled dually. The δ -functor L_*F is, by definition, the terminal object in the category of δ -functors G_* equipped with maps

$$\begin{array}{ccccccc} \cdots & \longrightarrow & G_1 M'' & \longrightarrow & G_0 M' & \longrightarrow & G_0 M & \longrightarrow & G_0 M'' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & & & FM' & \longrightarrow & FM & \longrightarrow & FM'' & & \end{array}$$

for every short exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

in \mathcal{C} ; we shall refer to such δ -functors as “ δ -functors over F .” The natural equivalence $F \circ F \simeq F$ gives us an equivalence of categories between the category of δ -functors over $F \circ F$ and the category of δ -functors over F . Each of these categories has a terminal object, since \mathcal{C} has enough projectives; hence the equivalence of categories gives us also an isomorphism between the two categories’ terminal objects, $L_*(F \circ F)$ and L_*F . \square

Lemma 4.3. *Let \mathcal{C} be an abelian category with enough projectives (resp. enough injectives). Let $\mathcal{C} \xrightarrow{F} \mathcal{C}$ be an additive functor equipped with a natural transformation $\text{id}_{\mathcal{C}} \xrightarrow{\eta} F$ (resp. $F \xrightarrow{\eta} \text{id}_{\mathcal{C}}$). Then η factors through the 0th left satellite functor L_0F (resp. η factors through the 0th right satellite functor R^0F).*

Proof. Assume \mathcal{C} has enough projectives; the dual case is handled dually. The identity functor is a δ -functor over F , and L_0F is, by definition, terminal in the category of δ -functors over F ; so the natural transformation $\text{id}_{\mathcal{C}} \rightarrow F$ factor as the composite

$$\text{id}_{\mathcal{C}} \rightarrow L_0F \rightarrow F.$$

\square

Proposition 4.4. The abelian category of L_0F -complete objects. *Let \mathcal{C} be an abelian category with enough projectives (resp. enough injectives). Let $\mathcal{C} \xrightarrow{F} \mathcal{C}$ be an additive functor equipped with a natural transformation $\text{id}_{\mathcal{C}} \xrightarrow{\eta} F$ (resp. $F \xrightarrow{\eta} \text{id}_{\mathcal{C}}$) such that the natural transformation $F \xrightarrow{F\eta} F \circ F$ (resp. $F \circ F \xrightarrow{F\eta} F$) is a natural equivalence of functors, and let \mathcal{D} be the full subcategory of \mathcal{C} generated*

by the objects X of \mathcal{C} such that the canonical map $X \rightarrow L_0F(X)$ (resp. the canonical map $R^0F(X) \rightarrow X$) is an isomorphism.

- (1) The functor L_0F (resp. the functor R^0F) is left adjoint (resp. right adjoint) to the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$.
- (2) If $M \xrightarrow{f} N$ is a morphism in \mathcal{D} , then the kernel (resp. cokernel) of f , computed in \mathcal{C} , is contained in \mathcal{D} and coincides with the kernel (resp. cokernel) of f computed in \mathcal{D} . The cokernel (resp. kernel) of f computed in \mathcal{D} is L_0F (resp. R^0F) applied to the cokernel (resp. kernel) of f computed in \mathcal{C} .
- (3) \mathcal{D} is an abelian category.
- (4) A morphism in \mathcal{D} is epic if and only if it is epic in \mathcal{C} . A morphism in \mathcal{D} is monic if and only if it is monic in \mathcal{C} .
- (5) The functor L_0F (resp. R^0F) sends projectives (resp. injectives) in \mathcal{C} to projectives (resp. injectives) in \mathcal{D} .
- (6) \mathcal{D} has enough projectives (resp. injectives).

Proof. Throughout, we assume \mathcal{C} has enough projectives; the dual case is handled dually.

- (1) We will write G for the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$. Let X be an object of \mathcal{C} and let Y be an object of \mathcal{D} . We construct maps

$$\begin{aligned} \mathrm{hom}_{\mathcal{C}}(X, GY) &\xrightarrow{\alpha} \mathrm{hom}_{\mathcal{D}}(L_0FX, Y), \\ \mathrm{hom}_{\mathcal{D}}(L_0FX, Y) &\xrightarrow{\beta} \mathrm{hom}_{\mathcal{C}}(X, GY), \end{aligned}$$

natural in X and Y , which are mutually inverse. Let $X \xrightarrow{f} GY$ be a map in \mathcal{C} ; then we let $\alpha(f)$ be the composite $\eta_Y^{-1} \circ L_0Ff$, where η_X, η_Y are the canonical maps

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \eta_X & & \downarrow \eta_Y \\ L_0FX & \xrightarrow{L_0Ff} & L_0FY \end{array} \quad .$$

Let $L_0FX \xrightarrow{g} Y$ be a map in \mathcal{D} ; then we let $\beta(g)$ be the composite $g \circ \eta_X$.

- (2) This is a corollary of Lemma 2.3.
- (3) That \mathcal{D} is an abelian category follows immediately from Prop. 2.5 and the previous parts of this proposition.
- (4) This is a corollary of Prop. 2.5 and the previous parts of this proposition.
- (5) Let P be a projective object of \mathcal{C} . Let $M \xrightarrow{f} N$ be an epimorphism in \mathcal{D} , and let $L_0FP \xrightarrow{g} N$ be a map in \mathcal{D} . Then we have the composite map $P \rightarrow L_0FP \xrightarrow{g} N$, and since P is projective in \mathcal{C} , the composite factors through f , so we have a morphism $P \xrightarrow{h} M$; since M is in \mathcal{D} , we know that h factors uniquely through the canonical morphism $P \rightarrow L_0FP$. The factor map $L_0FP \rightarrow M$ is a lift of g through f ; so L_0FP is projective in \mathcal{D} .
- (6) Let X be an object of \mathcal{D} , and let $P \xrightarrow{f} X$ be an epimorphism in \mathcal{C} with P projective in \mathcal{C} (such an epimorphism exists since \mathcal{C} has enough projectives). Then, since L_0F is a left adjoint and hence preserves colimits, $L_0FP \xrightarrow{L_0Ff}$

L_0FX is an epimorphism in \mathcal{D} and by the previous part of this proposition, L_0FP is projective in \mathcal{D} . The composite of L_0Ff with the isomorphism $L_0FX \xrightarrow{\cong} X$ is an epimorphism $L_0FP \rightarrow X$ in \mathcal{D} , and L_0FP is projective in \mathcal{D} ; so \mathcal{D} has enough projectives.

□

Definition 4.5. Let \mathcal{C} be an abelian category with enough projectives (resp. enough injectives). Let $\mathcal{C} \xrightarrow{F} \mathcal{C}$ be an additive functor equipped with a natural transformation $\text{id}_{\mathcal{C}} \xrightarrow{\eta} F$ (resp. $F \xrightarrow{\eta} \text{id}_{\mathcal{C}}$) such that the natural transformation $F \xrightarrow{F\eta} F \circ F$ (resp. $F \circ F \xrightarrow{F\eta} F$) is a natural equivalence of functors. Let $F\mathcal{C}$ be the full subcategory of \mathcal{C} generated by all objects X such that the natural map $X \xrightarrow{\eta(X)} F(X)$ (resp. $F(X) \xrightarrow{\eta(X)} X$) is an isomorphism. By a reflective abelian approximation to $F\mathcal{C}$ (resp. coreflective abelian approximation to $F\mathcal{C}$), we mean a subcategory \mathcal{D} of \mathcal{C} which is abelian and reflective (resp. coreflective) and such that η factors through the reflector (resp. coreflector) functor of \mathcal{D} , i.e., writing H for the forgetful functor $\mathcal{D} \xrightarrow{H} \mathcal{C}$, the natural transformation η factors as a composite of natural transformations

$$\begin{aligned} \text{id}_{\mathcal{C}} &\longrightarrow H \circ G \longrightarrow F \\ (\text{resp. } F &\longrightarrow H \circ G \longrightarrow \text{id}_{\mathcal{C}}), \end{aligned}$$

where $\text{id}_{\mathcal{C}} \rightarrow H \circ G$ (resp. $H \circ G \rightarrow \text{id}_{\mathcal{C}}$) is the natural transformation given by the adjointness of G and H .

We order the collection of reflective (resp. coreflective) abelian approximations to $F\mathcal{C}$ by inclusion: \mathcal{D} is less than or equal to \mathcal{E} if \mathcal{D} is contained in \mathcal{E} . (Note that this collection is not necessarily a set, or even a class.) By a best reflective abelian approximation to $F\mathcal{C}$ (resp. best coreflective abelian approximation to $F\mathcal{C}$) we mean a least element in this partially ordered collection.

Proposition 4.6. The category of L_0F -complete objects is the best reflective abelian approximation to the category of F -complete objects. Let \mathcal{C} be an abelian category with enough projectives (resp. enough injectives). Let $\mathcal{C} \xrightarrow{F} \mathcal{C}$ be an additive functor equipped with a natural transformation $\text{id}_{\mathcal{C}} \xrightarrow{\eta} F$ (resp. $F \xrightarrow{\eta} \text{id}_{\mathcal{C}}$) such that the natural transformation $F \xrightarrow{F\eta} F \circ F$ (resp. $F \circ F \xrightarrow{F\eta} F$) is a natural equivalence of functors, and let \mathcal{D} be the full subcategory of \mathcal{C} generated by the objects X of \mathcal{C} such that the canonical map $X \rightarrow L_0FX$ (resp. the canonical map $R^0FX \rightarrow X$) is an isomorphism. Let $F\mathcal{C}$ be the full subcategory of \mathcal{C} generated by all objects X such that the natural map $X \xrightarrow{\eta} FX$ (resp. $FX \xrightarrow{\eta} X$) is an isomorphism. Then \mathcal{D} is the best reflective (resp. coreflective) abelian approximation to $F\mathcal{C}$.

In other words, suppose \mathcal{E} is a subcategory of \mathcal{C} such that \mathcal{E} is an abelian category, such that the forgetful functor $\mathcal{E} \rightarrow \mathcal{C}$ has a left (resp. right) adjoint $F_{\mathcal{E}}$, and such that η factors through $F_{\mathcal{E}}$:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & FX \\ & \searrow & \nearrow \\ & F_{\mathcal{E}}X & \end{array}$$

$$\left(\begin{array}{ccc} X & \xleftarrow{\eta} & FX \\ & \searrow & \swarrow \\ & F_{\mathcal{E}}X & \end{array} \right) \text{ resp.}$$

Then the inclusion map $FC \hookrightarrow \mathcal{E}$ factors as a composite of inclusions $FC \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{E}$, i.e., \mathcal{E} contains \mathcal{D} and \mathcal{D} contains FC .

Proof. We deal with the homological case (i.e., \mathcal{C} has enough projectives); the cohomological case is handled dually.

Let X be an object of \mathcal{C} , let P_1, P_0 be projectives in \mathcal{C} , and let $P_1 \xrightarrow{g} P_0$ be a morphism in \mathcal{C} with cokernel X . Now $\text{coker } Fg \cong L_0F(X)$ is an element of \mathcal{C} , but we need to show that it is in \mathcal{E} ; the theorem follows immediately from this. We write h for the canonical map

$$L_0F(X) \xrightarrow{h} G_{\mathcal{E}}F_{\mathcal{E}}L_0F(X)$$

in \mathcal{C} , and we note that $L_0F(X)$ is in \mathcal{E} if and only if h is an isomorphism. That h is monic follows immediately from Prop. 2.5 together with our assumption that \mathcal{E} is an abelian category. The following argument proves that h is epic:

- The object $\text{coker } Fg$ of \mathcal{C} is the initial object in the category of objects Y of \mathcal{C} equipped with a map $F(P_0) \rightarrow Y$ such that the composite

$$F(P_1) \xrightarrow{Fg} F(P_0) \rightarrow Y$$

is zero.

- The object $G_{\mathcal{E}}F_{\mathcal{E}}\text{coker } Fg$ of \mathcal{E} is the initial object in the category of objects Y of \mathcal{E} equipped with a map $\text{coker } Fg \rightarrow Y$.
- Consequently, $G_{\mathcal{E}}F_{\mathcal{E}}\text{coker } Fg$ is the initial object in the category of objects Y of \mathcal{E} equipped with a map $F(P_0) \rightarrow Y$ such that the composite

$$F(P_1) \xrightarrow{Fg} F(P_0) \rightarrow Y$$

is zero.

- By definition, the morphism h is epic (in \mathcal{C}) if and only if, for any two morphisms $\phi, \psi : G_{\mathcal{E}}F_{\mathcal{E}}\text{coker } Fg \rightarrow Y$ in \mathcal{C} such that $\phi \circ h = \psi \circ h$, it is true that $\phi = \psi$.
- Consequently, the morphism h is epic (in \mathcal{C}) if and only if, for any object Y in \mathcal{E} equipped with two maps $\phi, \psi : F(P_0) \rightarrow Y$ such that the composites $\phi \circ (Fg)$ and $\psi \circ (Fg)$ are both zero and such that ϕ and ψ are equal when regarded as morphisms in \mathcal{C} , the morphisms ψ and ϕ are equal in \mathcal{E} . But this follows immediately from \mathcal{E} being a subcategory of \mathcal{C} ; so h is epic.

Hence h is an isomorphism, and $L_0F(X)$ is in \mathcal{E} . \square

In the following corollary, we require that I be a *finitely generated* ideal; this is so that the I -adic completion functor is idempotent. See [10] for an example of a commutative local ring R such that completion at the maximal ideal is not idempotent on the category of R -modules; this kind of bad behavior does not occur when one completes at a finitely-generated ideal.

Corollary 4.7. *Let R be a commutative ring and let I be a finitely-generated ideal of R , and let F denote the I -adic completion functor on the category of R -modules. Let L_0FC denote the category, studied in [3] and [6], of R -modules M such*

that $M \rightarrow L_0F(M)$ is an isomorphism. Then L_0Fc is the best reflective abelian approximation to the category of I -adically complete R -modules. This remains true if we consider a commutative flat Hopf algebroid (A, Γ) rather than a commutative ring R , a finitely generated invariant ideal I of (A, Γ) , and the category of Γ -comodules instead of the category of R -modules.

Furthermore, by Prop. 3.1, the category L_0Fc inherits a tensor product from the category of R -modules (or the category of Γ -comodules), and it is a closed symmetric monoidal category.

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